A NOTE ON THE CAFIERO CRITERION IN EFFECT ALGEBRAS

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Abstract. We give an alternative proof of a Cafiero type theorem for measures on effect algebras.

1. Introduction

In this note we want to give an altenative proof of the Cafiero theorem valid for measures on effect algebras as contained in (cf. [1]). Avallone reduced the proof to the classical case using techniques elaborated in [11]; we here give a direct proof imitating de Lucia and Cavaliere's paper (see [7]).

Effect algebras (alias D-posets) have been independently introduced in 1994 by D. J. Foulis and M. K. Bennett in [3] and by F. Chovanek and F. Kopka in [5] for modelling unsharp measurement in a quantum mechanical system. They are a generalization of many structures which arise in Quantum Physics [8] and in Mathematical Economics [6, 4], in particular they are a generalization of orthomodular posets and MV-algebras and therefore of Boolean algebras.

2. Preliminaries

Definition 2.1. Let $(L, \leq)$ be a poset with a smallest element 0 and a greatest element 1 and let $\ominus$ be a partial operation on L such that $b \ominus a$ is defined if and only if $a \leq b$ and for all a, b, $c \in L$:

If $a \leq b$ then $b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$

If $a \leq b \leq c$ then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

Then $(L, \leq, \ominus)$ is called a *difference poset (D-poset* for short), or a *difference lattice (D-lattice* for short) if L is a lattice.

One defines in L a partial operation $\oplus$ as follows:

$a \oplus b$ is defined and $a \oplus b = c$ if and only if $c \ominus b$ is defined and $c \ominus b = a$.

The operation $\oplus$ is well-defined by the cancellation law [8, page 13] ($a \leq b, c$ and $b \ominus a = c \ominus a$ implies $b = c$), and $(L, \oplus, 0, 1)$ is an effect algebra (see [8, Theorem 1.3.4]), that is the following conditions are satisfied for all $a, b, c \in L$:

If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$;

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If $b \oplus c$ is defined and $a \oplus (b \oplus c)$ is defined, then $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$;

There exists a unique $a' \in E$ such that $a \oplus a'$ is defined and $a \oplus a' = 1$;

If $a \oplus 1$ is defined, then $a = 0$.

We say that a and b are orthogonal if $a \leq b'$ and we write $a \perp b$. Therefore $a \oplus b$ is defined if and only if $a \perp b$, and in this case $a \oplus b = (a' \ominus b)'$ by [8, Lemma 1.2.5].

From now on, let L be a D-lattice.

In the sequel we deal with functions defined on L with values in a topological space (S, τ).

Definition 2.2. A map $\mu : L \to S$ is called a *measure* if $\mu(a \oplus b) = \mu(a) + \mu(b)$ whenever a, $b \in L$ are orthogonal.

Classical measures on Boolean algebras are example of measures on effect algebras.

We employ the notation:

Notation 2.3. Let $e \in S$ be a point. We denote by $\mathcal{M}$ the collection of all functions $\mu : L \to S$ such that $\mu(0) = e$ and by $\tau[e]$ a fundamental system of neighbourhoods of e. Moreover $M \in I_\infty(\mathbb{N})$ means M is an infinite subset of $\mathbb{N}$.

Definition 2.4. A function of $\mathcal{M}$ is said exhaustive whenever $\lim_k \mu(a_k) = e$ for every orthogonal sequence (a_k) as well as a sequence (μ_n) of elements of $\mathcal{M}$ is said to be uniformly exhaustive if $\lim_k \mu_n(a_k) = e$, uniformly with respect to $n \in \mathbb{N}$, for any orthogonal sequence (a_k) in L. For any function $\mu \in \mathcal{M}$ we put $\tilde{\mu}(a) := \{\mu(b) : \ b \in L, \, b \leq a\}$ for every $a \in L$.

Lemma 2.5. *If $\mu \in \mathcal{M}$ is exhaustive and (a_k) is an orthogonal sequence in L, then for every $P \in I_\infty(\mathbb{N})$ and every $U \in \tau[0]$, there exists $M \in I_\infty(P)$ such that $\bigoplus_{k \in M} a_k$ exists in L and $\tilde{\mu}(\bigoplus_{k \in M} a_k) \subseteq U$.*

Proof. The proof is straightforward. ☐

In [1] Avallone introduced the following definition:

Definition 2.6. We say that L satisfies the D-subsequential completeness property (D-SCP, for short) if for every orthogonal sequence (a_n) in L there is $M \in I_\infty(\mathbb{N})$ such that $\bigoplus_{n \in M} a_n$ exists.

Lemma 2.7. *Let L with D-SCP property.If μ_n is a sequence of exhaustive elements of $\mathcal{M}$, then, for every $U \in \tau[0]$, any orthogonal sequence (a_k) in L admits a subsequence a_{k_i} such that $\bigoplus_{i \in \mathbb{N}} a_{k_i}$ in L and $\tilde{\mu}_{k_j}(\bigoplus_{i > j} a_{k_i}) \subseteq U$ for every $j \in \mathbb{N}$.*

Proof. Let $U \in \tau[0]$ and let (a_k) be an orthogonal sequence in L.

Since μ_1 is exhaustive, by Lemma 2.5, there exists $M_0 \in I_\infty(\mathbb{N} \setminus \{1\})$ such that $\bigoplus_{k \in M_0} a_k$ exists in L and $\tilde{\mu}_1(\bigoplus_{k \in M_0} a_k) \subseteq U$.

Let $k_1 := \min M_0$. By Lemma 2.5 again, there exists $M_1 \in I_\infty(M_0 \setminus \{k_1\})$ such that $\bigoplus_{k \in M_1} a_k$ exists in L and $\tilde{\mu}_{k_1}(\bigoplus_{k \in M_1} a_k) \subseteq U$.

Going on by induction, one can determine an increasing sequence (k_m) in $\mathbb{N}$ and a decreasing sequence (M_m) in $I_\infty(\mathbb{N})$ such that for every $m \in \mathbb{N}$: $\bigoplus_{k \in M_m} a_k$ exists in L and $\tilde{\mu}_{k_m}(\bigoplus_{k \in M_m} a_k) \subseteq U$ with $k_m \notin M_m$.

By the D-SCP property, the orthogonal sequence (a_{k_m}) admits a subsequence $(a_{k_{m_i}})$ such that there exists in L the supremum $\bigoplus a_{k_{m_i}}$.

Since for every $j \in \mathbb{N}$ it holds that $\bigoplus_{i>j} a_{k_{m_i}} \leq \bigoplus_{k \in M_{m_j}} a_k$ one has that

$$\tilde{\mu}_{k_{m_j}}(\bigoplus_{i>j} a_{k_{m_i}}) \subseteq U \quad \forall j \in \mathbb{N},$$

which ends the proof. □

Definition 2.8. The function μ in M is called quasi-triangular whenever for every U in $\tau[0]$ there exists $V(U) \in \tau[0]$ such that it holds

$a \perp b$, $\mu(a) \in V$, $\mu(b) \in V \Rightarrow \mu(a \bigoplus b) \in U$

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The functions (μ_n) in M are called uniformly quasi-triangular whenever for every U in $\tau[0]$ there exists $V(U) \in \tau[0]$ such that for all $n \in \mathbb{N}$ it holds

$a \perp b$, $\mu_n(a) \in V$, $\mu_n(b) \in V \Rightarrow \mu_n(a \bigoplus b) \in U$

$a \perp b$, $\mu_n(a) \in V$, $\mu_n(a \bigoplus b) \in V \Rightarrow \mu_n(b) \in U$

Quasi-triangular functions generalize functions $\mu \colon L \to [0, +\infty]$ satisfying

$$|\mu(a \oplus b) - \mu(a)| \leq \mu(b)$$

for orthogonal elements $a, b \in L$. Such functions were considered in the classical context and are called triangular by some authors.

Lemma 2.9. *Let L with D-SCP property. Given a sequence (μ_n) of exhaustive and uniformly quasi-triangular elements of $\mathcal{M}$, if*

for every $U_o \in \tau[0]$ and for every orthogonal sequence (b_k) in L there exists $k_0 \in \mathbb{N}$ such that $\{n \in \mathbb{N} : \mu_n(b_{k_0}) \notin U_0\}$ is finite,

then for every $U \in \tau[0]$ and every orthogonal sequence (a_k) in L such that $\mu_k(a_k) \notin U$ $\forall k \in \mathbb{N}$, there exist an increasing sequence (k_m) in $\mathbb{N}$ and $M \in I_\infty(\mathbb{N})$ such that $\exists \bigoplus_{j \in M} a_{k_j}$ and $\tilde{\mu}_{k_m}(\bigoplus_{j \in M} a_{k_j}) \notin V(U)$ $\forall m \in \mathbb{N}$.

Proof. Let $U \in \tau[0]$ be given. Since the μ_n's are uniformly quasi-triangular, one can consider $V_0 := V(U)$ and $V_n = V_{n-1} \cap V(V_{n-1})$ like in Definition 2.8.

By Lemma 2.7, taking subsequences if needed, on has that

$$\tilde{\mu}_m(\bigoplus_{k>m} a_k) \subseteq V_1 \ \forall m \in \mathbb{N} \quad (*).$$

Moreover, from assumptions there exist two natural numbers k_1 and n_1 such that

$$\mu_n(a_{k_1}) \in V_2 \ \forall n > n_1$$

as well as there exist n_2 and k_2 such that

$$k_2 > \max\{k_1, n_1\} \ n_2 > n_1 \ \mu_n(a_{k_2}) \in V_3 \ \forall n > n_2.$$

Thus, by induction, one can construct two strictly increasing sequence (k_j) and (n_j) such that

$$k_j > n_{j-1} \text{ and } \mu_{k_m}(a_{k_j}) \in V_{j+1} \ \forall m > j \quad (**).$$

Since L has the D-SCP, there exists $M \in I_\infty(\mathbb{N})$ such that there exists $\bigoplus_{j\in M} a_{k_j}$. Moreover one infers from $(*)$ that $\tilde{\mu}_{k_m}(\bigoplus_{j>mj\in M} a_{k_j}) \subseteq V_1 \ \forall m \in \mathbb{N}$, and from $(**)$ that

$$\mu_{k_m}(\bigoplus_{j<m, j\in M} a_{k_j}) \in V_1 \ \forall m \in \mathbb{N}.$$

Hence, by the uniform quasi-triangularity of the μ_n, it follows that

$$\mu_{k_m}(\bigoplus_{j\neq m, j\in M} a_{k_j}) \in V_0 \ \forall m \in \mathbb{N},$$

so by hypothesis one can establish that $\tilde{\mu}_{k_m}(\bigoplus_{j\in M} a_{k_j}) \notin V_0 \ \forall m \in \mathbb{N}$, as desired. □

3. Cafiero criterion

Now we are able to proof our main result.

Theorem 3.1. *Let L with D-SCP property. Let (μ_n) be a sequence ofexhaustive and uniformly quasi-triangular functions. Then (μ_n) is uniformly exhaustive if and only if the following condition holds*

for every $U \in \tau[0]$ and every orthogonal sequence (a_k) there exist $k_0, n_0 \in \mathbb{N}$ such that $\mu_n(a_{k_0}) \in U \ \forall n \geq n_0$.

Proof. The necessity of the condition is trivial.

For the sufficiency, we argue by contradiction. Let us assume, by passing to a subsequence if necessary, that there exists an orthogonal sequence (a_k) such that

$$\mu_n(a_n) \notin U_0 \ \forall n \in \mathbb{N}.$$

Let (P_k) be a disjoint sequence in $I_\infty(\mathbb{N})$ whose elements cover $\mathbb{N}$. By 2.9 for every $k \in \mathbb{N}$ there exists $M_k \in I_\infty(P_k)$ such that $\exists \bigoplus_{j\in M_k} a_j$ and the set $\{n \in \mathbb{N} : \mu_n(\bigoplus_{j\in M_k} a_j) \notin V(U_0)\}$ is infinite.

The above construction guarantees that the sequence $(\bigoplus_{j\in M_k} a_j)_k$ is orthogonal and that for every $k \in \mathbb{N}$ the set $\{n \in \mathbb{N} : \mu_n(\bigoplus_{j\in M_k} a_j) \notin V(U_0)\}$ is infinite, but this contradicts the hypothesis. □

Theorem 3.2. *Let L with D-SCP property. Let (μ_n) be a sequence of exhaustive and uniformly quasi-triangular elements of $\mathcal{M}$. If (μ_n) converges pointwise to a exhaustive element μ of $\mathcal{M}$, then (μ_n) is uniformly exhaustive.*

Proof. Let us consider an open element U of $\tau[0]$ and an orthogonal sequence (a_k). Since μ is exhaustive, there exists $k_0 \in \mathbb{N}$ such that $\mu(a_k) \in U$ for every $k \geq k_0$. Thus the result comes applying 3.1. □

Theorem 3.2 furnishes an alternative proof of [1, Theorem 4.3].

References

[1] A. Avallone Cafiero and Nikodym boundedness theorems in effect algebras. *Ital. J. Pure Appl. Math.* 20 (2006), 203–214.

[2] F. Cafiero Sulle famiglie di funzioni additive d'insieme, uniformemente continue. *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat.* (8) 12, (1952). 155–162.

[3] M. K. Bennett and D. J. Foulis. Effect algebras and unsharp quantum logics. Special issue dedicated to Constantin Piron on the occasion of his sixtieth birthday. *Found. Phys.* 24(10):1331–1352, 1994.

[4] D. Butnariu and P. Klement. *Triangular Norm-based Measures and Games with Fuzzy Coalitions.* Kluwer Academic Publishers, Dordrecht, 1993.

[5] F. Chovanek and F. Kopka. D-posets. *Math. Slovaca* 44(1):21–34, 1994.

[6] Epstein and Zhang. Subjective probabilities on subjectively unambiguous events. *Econometrica*, 69(2):265–306, 2001.

[7] P. de Lucia and P. Cavaliere . The Cafiero criterion on a Boolean ring. *Rend. Accad. Sci. Fis. Mat. Napoli* (4) 75 (2008), 1–5.

[8] A. Dvurecenskij and S. Pulmannová, New trends in quantum structures. Kluwer Academic Publishers, Bratislava (2000).

[9] F. Ventriglia, Cafiero and Brooks-Jewett theorems for Vitali spaces. *Ric. Mat.* 56 2 (2007), 209–216.

[10] F. Ventriglia, Cafiero theorem for k-triangular functions on an orthomodular lattice *Rend. Accad. Sci. Fis. Mat. Napoli* (4) 75 (2008), 45–52.

[11] H. Weber Compactness in spaces of group-valued contents, the Vitali-Hahn-Saks theorem and Nikodm's boundedness theorem. *Rocky Mountain J. Math.* 16 2 (1986), 253–275.

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