Programming Concepts and Methods
M. Broy & C.B. Jones (Editors)
Elsevier Science Publishers B.V. (North-Holland)

REASONING ABOUT INTERPRETATIONS IN QUALITATIVE λ-MODELS

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1. INTRODUCTION

Qualitative Domains are particular partially ordered sets introduced by Girard [6],[7] as an alternative scenario to Scott Domains for the semantics of lambda calculi. The main difference between these two notions of domain lies in the function space construction. In fact, if D and E are two qualitative domains, the qualitative function space from D to E consists of a proper subset of all Scott continuous functions from D to E endowed by an order relation which is strictly included in the pointwise ordering. Elements of the qualitative function space were called stable functions by Berry [4], who first introduced them in the context of sequential functions. The order relation on stable functions induces a finer notion of approximation than the one induced by the extensional pointwise ordering. Recently qualitative domains have been utilized by Girard for the coherent semantics of Linear Logic [8].

The interpretations of λ-terms in a given qualitative λ-model are particularly difficult to compute and visualize. It is therefore desirable to build a formal system for correctly reasoning about them. In this paper we show how to define such a formal system for an interesting class of qualitative λ-models. Actually, when a qualitative λ-model M of this class has a finite inductive definition, our formal system turns into a type assignment system $\vdash_M$ for λ-terms; the set of types of $\vdash_M$ being isomorphic to the set of atoms of M. It is then possible to derive a type σ for a λ-term M if and only if the corresponding atom belongs to the interpretation of M. A similar relation between Scott's D_∞-λ-models and intersection type assignment systems for λ-calculus was studied in [2], [5], [10]. See also [1] for related work.

Describing qualitative λ-models through a type assignment system provides a deep insight into the structure of qualitative domains themselves. Moreover this technique makes possible to apply standard methods in proof theory to the study of the fine structure of the models. In this paper, for instance, we derive an Approximation Theorem for a particular model from a normalization property of the type derivation system (i.e., we show that the interpretation of a term is the union of the interpretations of its syntactical approximants).

Quite frequently the type assignment systems $\vdash_M$ that we introduce are significant in themselves, besides their connection with M. The most striking feature of a system $\vdash_M$

Work partially supported by the EEC "Project Stimulation ST2J/0374/C(EDB): Lambda Calcul Type'" and MPI 40%.

is that the structural *weakening rule* fails for it, or equivalently that $\vdash_M$ has a relevant assumption discipline. I.e., if $\Gamma \vdash_M M:\sigma$ and $\{x:\alpha\} \subset \Gamma$, then x occurs free in M. This is due to the fact that the meaning of the consequence relation axiomatized by $\vdash_M$ over typing judgements, i.e. $\{x_1:\alpha_1, \ldots, x_n:\alpha_n\} \vdash_M M:\sigma$ is that $\{x_1:\alpha_1, \ldots, x_n:\alpha_n\}$ is a *minimal* set of assumptions under which $M:\sigma$ is derivable. This is in sharp contrast to what happens in the intersection type assignment systems, where if $\Gamma \vdash_M M:\sigma$ then, for every x and α, $\Gamma \cup \{x:\alpha\} \vdash_M M:\sigma$.

In Section 2 of this paper we review the basic definitions and results in the theory of qualitative domains and define three λ-models ***Q***, ***P*** and ***S***. In Section 3 we define the notion of a qualitative λ-structure (D,i). We introduce a formal system $S_{(D,i)}$ and we show that it is sound and complete for deriving approximation judgements between interpretations in (D,i). In Section 4 we give finite inductive presentations of the models ***Q***, ***P*** and ***S*** and we tailor the system $S_{(D,i)}$ to each of these, thus producing three type assignment systems $\vdash_Q$, $\vdash_S$, $\vdash_P$. Finally in Section 5 we give a few examples of how the system $\vdash_M$ can be used to analyze the fine structure of ***M***. In particular we prove an Approximation Theorem for ***Q***, using a normalization property of the type assignment system and show that the theory of the model ***P*** does not equate all unsolvable terms. The results concerning the model ***Q*** appearing in this paper are proved using a different technique also in [11].

Finally, two remarks are in order.

In this paper we are concerned with the untyped λ-calculus. Clearly this is one of the most intriguing and difficult calculi. Our results subsume results for models of typed λ-calculi, based on qualitative domains.

Moreover, the techniques outlined in this paper can be generalized to the coherent semantics for languages based on Girard's linear logic.

Throughout the paper we assume the reader familiar with the basic notions and notations of λ-calculus as given in [3].

2. QUALITATIVE λ-MODELS.

In this section we recall some definitions concerning qualitative domains and we review the theory of qualitative λ-models [6]. Finally we define three examples of qualitative λ-models : ***Q***, ***P*** and ***S***.
We will denote standard set theoretic inclusion (not necessarily strict), with $\subset$.

DEFINITION 1.
i) A *qualitative domain* D is a set of sets such that:
   - $\emptyset \in D$
   - D is closed under directed unions: i.e.,
     if $\forall i \in I\ (A_i \in D)$ and $\forall i,j \in I\ \exists k \in I\ (A_i \cup A_j \subset A_k)$ then $\cup_{i\in I} A_i \in D$;
   - if $a \in D$ and $b \subset a$, then $b \in D$;
ii) a qualitative domain D is *binary* if moreover:
   - if $d \subset \cup D$ and $d \notin D$ then there are $a,b \in d$ such that $\{a, b\} \notin D$.

Binary qualitative domains are known as coherent spaces in linear logic. These structures provide a denotational semantics, in the tradition of Heyting and Scott, for proofs in full linear logic.

Elements of the set $\cup D = \{ z \mid z \in a \in D \}$ are called *atoms* of D. This set will also be denoted with $|D|$. $a \in D$ is *finite* if and only if $\{z \mid z \subset a\}$ is finite. Two elements of D, say a and b, are *compatible* if and only if $a \cup b \in D$.

DEFINITION 2.
Let D and D' be two qualitative domains;

i) a function $F: D \rightarrow D'$ is *stable* if and only if:
   1) if $a \subset b \in D$ then $F(a) \subset F(b)$;
   2) if $\{ a_i \mid i \in I\}$ is a not empty directed set then $F(\cup_{i\in I} a_i) = \cup_{i\in I} F(a_i)$;
   3) if $a \cup b \in D$ then $F(a \cap b) = F(a) \cap F(b)$;
ii) let $F: D \rightarrow D'$ be a stable function. The *trace* of F is:
   $Tr(F) = \{(a,z) \mid a$ is a finite element of D, $z \in |D'|$, $z \in F(a)$ and $z \notin F(a')$ for all $a' \subset a$ but $a \neq a'\}$;
iii) stable functions can be ordered according to the order relation $\sqsubseteq_s$ introduced by Berry [4] defined as follows: let F, G: $D \rightarrow D'$ be two stable functions:
   $F \sqsubseteq_s G$ if and only if $\forall a,b \in D$ ($a \subset b$ implies $F(a) = F(b) \cap G(a)$).

One can easily check that if $F \sqsubseteq_s G$ then $\forall a \in D$, $F(a) \subset G(a)$.
The following proposition illustrates two useful properties of stable functions under Berry's order relation.

PROPOSITION 3.
i) If F is a constant function and $G \sqsubseteq_s F$ then G is also a constant function.
ii) Let $F: D \rightarrow D'$ be a stable function. If both (a,z) and (b,z) belong to Tr(F) then a and b are not compatible.

PROOF.
i) Assume $G \sqsubseteq_s F$ and let $G(a) \neq G(b)$. Without loss of generality we can assume $a \subset b$. Then $G(a) = F(a) \cap G(b)$. But this is impossible since $G(b) \subset F(a)$.
ii) Let a and b be compatible. By definition of stable function, $F(a \cap b) = F(a) \cap F(b)$, so $z \in F(a \cap b)$. But, by definition of trace, $z \in F(a \cap b)$ implies there exists $d \subset (a \cap b)$ such that $(d,z) \in Tr(F)$ and hence $(a,z) \notin Tr(F)$ and $(b,z) \notin Tr(F)$.
□

Proposition 3.i) implies that $\sqsubseteq_s$ is *strictly* finer than the pointwise order relation. In fact, let $f_{a,b}$ denote the step function $\boldsymbol{\lambda}x.$ *if* $a \subset x$ *then* b *else* $\emptyset$. Then $f_{a,b}$ is clearly stable and pointwise smaller than the constant function $\lambda x.b$. But it is not the case that $f_{a,b} \sqsubseteq_s \lambda x.b$.

The construction of the qualitative function space is made possible by the following theorem. See [6] for a proof.

REPRESENTATION THEOREM.
Let D, D' be two qualitative domains.
i) A stable function $F: D \rightarrow D'$ is completely determined by its trace Tr(F) in the following sense: $F(a) = \{z \mid (d,z) \in Tr(F)$ and $d \subset a\}$.
ii) Let $[D \rightarrow_s D']$ be the set of the traces of all stable functions from D to D'. $[D \rightarrow_s D']$ is a qualitative domain, and it is isomorphic to the set of all stable functions from D to D', ordered by $\sqsubseteq_s$. Hence $G \sqsubseteq_s F$ if and only if $Tr(G) \subset Tr(F)$.

As remarked earlier, qualitative domains can be used to provide a denotational semantics for programming languages. We will show how to do this for the untyped lambda calculus Λ. Namely, we will introduce a suitable category **BQual**, where we will be able to manufacture qualitative λ-structure and qualitative λ-models.

DEFINITION 4.
i) Let D and D' be qualitative domains. A *qualitative morphism* from D to D' is an injective function $f: |D| \rightarrow |D'|$ such that for all $a_1, \ldots, a_n \in |D|$, $\{a_1, \ldots, a_n\} \in D$ if and only if $\{f(a_1), \ldots, f(a_n)\} \in D'$.
ii) **BQual** is the category with binary qualitative domains as objects and with qualitative morphisms as morphisms.

DEFINITION 5.

i) A *qualitative λ-structure* is a pair (D, i) where D is a binary qualitative domain and $i : |R| \to |D|$ is an injective function, where $R \subset [D \to_s D]$ is a qualitative domain. R is said to be the domain of *representable functions*.

ii) The *interpretation* of λ-terms in a qualitative λ-structure (D, i) is a function $[\![\ ]\!] : \Lambda \to Env \to D$ inductively defined as follows:

$[\![x]\!]\rho = \rho(x)$

$[\![MN]\!]\rho = H_i([\![M]\!]\rho)\ [\![N]\!]\rho$

$[\![\lambda x.M]\!]\rho = K_i(\lambda d \in D.\ [\![M]\!]\rho[d/x])$

where:

$H_i(d) = \{a \mid i(a) \in d\}$ and $K_i(e) = \{i(a) \mid a \in e\}$

$Env = \{\rho \mid \rho : Variables \to D\}$ and $\rho[d/x]$ denote the function:

$\rho[d/x](y) =$ *if* $x \equiv y$ *then* d *else* $\rho(y)$.

iii) Environments are partially ordered componentwise, i.e., $\rho \le \rho'$ if and only if $\forall x.\rho(x) \subset \rho'(x)$.

Similarly two environments ρ and ρ' are said to be compatible if they are componentwise compatible.

The interpretation function satisfies the following properties; see [6] for a proof:

PROPOSITION 6.

Let (D, i) be a qualitative λ-structure.

i) $H_i \circ K_i \subset Id_{[D \to_s D]}$, and hence $[\![(\lambda x.M)N]\!]\rho \subset [\![M[N/x]]\!]\rho$. The β) rule holds if and only if $H_i \circ K_i$ is an isomorphism.

ii) $K_i \circ H_i \subset Id_D$, and hence $[\![(\lambda x.Mx)]\!]\rho \subset [\![M]\!]\rho$ provided x does not occur free in M.

The η) rule holds if and only if $K_i \circ H_i$ is an isomorphism.

The above proposition motivates the following important definition.

DEFINITION 7.

A qualitative λ-structure (D, i) is a *qualitative λ-model* if $H_i \circ K_i = Id_{[D \to_s D]}$, and it is an *extensional qualitative λ-model* if moreover $K_i \circ H_i = Id_D$.

One can easily check that a qualitative λ-model is a λ-model according to the standard definition given in [9].

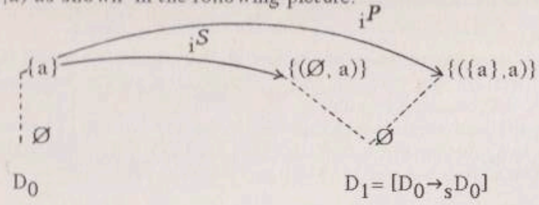
In **BQual** qualitative λ-models can be built using limit construction as in [6]. In fact Girard [6] showed that the stable function space constructor $[.\to_s.]$, the cartesian product constructor × and the coalesced sum constructor + are functors in **BQual**. Moreover these functors are "well behaved" in the sense that recursive equations in **BQual** written using them can be solved using standard limit constructions.

Our notion of qualitative λ-structure and qualitative λ-models are not as general as those introduced by Girard in [6]. Namely we restricted the original notions to the category BQual. We could have dealt with Girard's notions, but this would have lead to inessential and tediuos complications in the following sections.

In this paper we discuss in detail three particular qualitative λ-models. The first model $\mathcal{Q} \equiv (D_{\mathcal{Q}}, i_{\mathcal{Q}})$ is the "minimal" solution, up to isomorphism, of the recursive equation in **BQual**: $D = P(\omega) \times [D \to_s D]$. It is defined as follows: $D_{\mathcal{Q}} = \lim_{n \ge 0}(D_n, i_n)$, where $D_0 = \{\varnothing\}$, $D_{n+1} = P(\omega) \times [D_n \to_s D_n]$ and the initial morphism i_0 from D_0 to D_1 is the empty function. $i_{\mathcal{Q}}$ is naturally induced by i_0, namely $i_{\mathcal{Q}} = \lambda x \in |[D \to_s D]|.\ (\varnothing, x)$.

The other two models are both solutions of the equation $D = [D \to_s D]$. Let $D_0 = \{\varnothing, \{a\}\}$ and $D_{n+1} = [D_n \to_s D_n]$. We can define two morphisms from D_0 to D_1, namely $i^{\mathcal{S}}(a) = (\varnothing, a)$

and $i^{\mathcal{P}}(a) = (\{a\}, a)$ as shown in the following picture:



Let $D_{\mathcal{S}}$ be $\lim_{n \ge 0}(D_n, i_n)$ obtained by taking $i_0 = i^{\mathcal{S}}$ and let $D_{\mathcal{P}}$ be the limit obtained by taking $i_0 = i^{\mathcal{P}}$. We will call $\mathcal{S}$ the model $(D_{\mathcal{S}}, i_{\mathcal{S}})$, where $i_{\mathcal{S}} = i_\infty$, and $\mathcal{P}$ the model $(D_{\mathcal{P}}, i_{\mathcal{P}})$, where $i_{\mathcal{P}} = i_\infty$. The names $\mathcal{S}$ and $\mathcal{P}$ are reminiscent of the two analogue solutions which were given respectively by Scott [14] and Park [12] to the equation $D = [D \to D]$ in the category of Scott domains.

We will end this section by pointing out a property of the interpretation function, which will be useful in the sequel.

PROPOSITION 8.

Let ρ and ρ' be two environments and let M be a λ-term.

i) If $\rho \le \rho'$ then $[\![M]\!]\rho \sqsubseteq [\![M]\!]\rho'$.

ii) If ρ and ρ' are compatible then also $[\![M]\!]\rho$ and $[\![M]\!]\rho'$ are compatible.

PROOF.

Both points can be easily proved by induction on the structure of M.

□

3. FORMAL SYSTEM FOR REASONING ABOUT INTERPRETATIONS IN QUALITATIVE λ-MODELS.

As we remarked in the introduction, it is particularly difficult to reason about the interpretations of λ-terms in a given qualitative λ-structure. It is therefore desirable to build a formal system for correctly reasoning about these denotations.

A qualitative λ-structure (D,i) is completely described once we have specified three objects: |D|, the compatibility predicate over |D|×|D|, and the function i. Hence the interpretation of a term in (D,i) can be completely characterized once we have specified the atoms of D belonging to it. The problem we want to solve can therefore be phrased in the following way:

PROBLEM.

Given a qualitative λ-structure (D, i), build a formal system ⊢ for establishing judgements of the form M:a under a set of assumptions Γ; the meaning of Γ⊢ M:a being that the atom a belongs to the interpretation of M in (D,i) when the free variables of M are interpreted as in Γ.

In case we have a finite inductive definition of (D, i), we can refer to the atoms of D as *types* and to the formal system as a type assignment system.

The analogous problem for Scott D_∞-λ-models is discussed in several papers, e.g.[5] and [10]. The language for describing finite elements of Scott Domains is the so called *intersection* type language [2]. The main result is: for every Scott D_∞-λ-model it is possible to define an appropriate intersection type assignement system which satisfies the requirements of the problem above.

From now onwards, throughout all this section, fix a qualitative λ-structure (D, i), and assume that we are given a notation for the elements of |D|.

We will denote with $(\{a_1,\dots,a_n\}, a)$ where a, a_j are notations for elements of |D|, the unique atom of $\|[D \to_s D]\|$ in the trace of the step-function $\lambda x.\ \mathit{if}\ (\forall j.\ a_j \in x\ )\ \mathit{then}\ a\ \mathit{else}\ \emptyset$, and with $i(\,(\{a_1,\dots,a_n\},a))$ the corresponding atom in D, when it exists.

Define a *context* Γ as a set of pairs x:a, where x is a variable and $a \in |D|$. As explained above the formal system we want to build should manipulate judgements of the form $\Gamma \vdash M{:}a$. We want the following to hold:

$\Gamma \vdash M{:}a$ is derivable if and only if $a \in [\![M]\!]\rho$ where $\rho(x) = \{b \mid x{:}b \in \Gamma\}$.

The definition of the interpretation function (see Definition 5) suggests a possible set of rules which unfortunately is not sound. More precisely the rule:

$$\text{(abstr)}\ \frac{\Gamma \cup \{x{:}a_1,\dots,x{:}a_n\} \vdash M{:}a \quad b = i((\{a_1,\dots,a_n\},a))}{\Gamma \vdash \lambda x.M{:}b}$$

is incompatible with the (semantically obvious) projection rule:

$$\text{(proj)}\ \frac{x{:}a \in \Gamma}{\Gamma \vdash x{:}a}$$

or with the structural rule of weakening of premises in Γ, i.e., with the rule:

$$\text{(weak)}\ \frac{\Gamma \vdash M{:}a}{\Gamma \cup \{x{:}b\} \vdash M{:}a.}$$

In fact, using these rules, one can easily derive that a non constant function "approximates" a constant function, say $\lambda x.y$, thus contradicting Proposition 3.i). For istance:

$$\text{(abstr)}\ \dfrac{\text{(proj)}\ \dfrac{y{:}b \in \{x{:}a,\ y{:}b\}}{\{x{:}a,\ y{:}b\} \vdash y{:}b} \qquad c = i((\{a\},b))}{\{y{:}b\} \vdash \lambda x.y{:}c}$$

and hence $\lambda x.\mathit{if}\ a \in x\ \mathit{then}\ b\ \mathit{else}\ \emptyset \sqsubseteq_s \lambda x.y$. This is in sharp contrast to what happens in intersection type assignment systems. In fact, in the latter, the arrow type constructor behaves like intuitionistic implication.

In order to capture correctly the behaviour of traces in qualitative λ-structures, we need a formal system with a relevant discipline of assumptions.

Let Γ be a context. We will denote with $\mathrm{dom}(\Gamma)$ the set $\{x \mid \exists a.\ x{:}a \in \Gamma\}$.

THE FORMAL SYSTEM $S_{(D,i)}$

$S_{(D,i)}$ consists of the following rules:

$$(1)\ \frac{}{\{x{:}a\} \vdash x{:}a}$$

$$(2)_{(n \geq 0)}\ \frac{\Gamma \vdash M{:}b \quad \{\Gamma_j \vdash N{:}a_j\}_{1 \leq j \leq n} \quad b = i((\{a_1,\dots,a_n\},a))}{\Gamma \cup (\cup_{1 \leq j \leq n} \Gamma_j) \vdash MN{:}a}$$

$$(3)_{(n \geq 0)}\ \frac{\Gamma \cup \{x{:}a_1,\dots,x{:}a_n\} \vdash M{:}a \quad \{a_1,\dots,a_n\} \in D \quad x \notin \mathrm{dom}(\Gamma) \quad b = i((\{a_1,\dots,a_n\},a))}{\Gamma \vdash \lambda x.M{:}\,b}$$

Note that both rules (2) and (3) are parametric in n, as the notations $(2)_{(n \geq 0)}$ and $(3)_{(n \geq 0)}$ indicate. In particular the instances of rule (2) and (3) for n=0 have the simple form:

$$(2)\ \frac{\Gamma \vdash M{:}b \quad b = i((\emptyset,a))}{\Gamma \vdash MN{:}a}$$

$$(3)\ \frac{\Gamma \vdash M{:}a \quad x \notin \mathrm{dom}(\Gamma) \quad b = i(\,(\emptyset,a))}{\Gamma \vdash \lambda x.M{:}\,b.}$$

We assume that the judgements $\{a_1,\dots,a_n\} \in D$ and $b = i((\{a_1,\dots,a_n\},a))$ occurring in the premises of rules (2) and (3) can be established using suitable formal systems $\in_D$ and $=_i$ which encode, possibly finitely, the compatibility relation on |D| and the graph of i.

We will now show that $S_{(D,i)}$ is the correct solution to the problem stated above.

A context Γ is said to be *consistent* if and only if $\{x{:}a,\ x{:}b\} \subset \Gamma$ implies a and b are compatible. Moreover two contexts are said to be *consistent* if and only if their union is consistent.

SOUNDNESS THEOREM.

Let (D, i) be a qualitative λ-structure, and let M be a λ-term. Then $\Gamma \vdash M{:}a$ and Γ consistent imply that $a \in [\![M]\!]\rho_\Gamma$ in (D, i), where $\rho_\Gamma(x) = \{b \mid x{:}b \in \Gamma\}$. Moreover ρ_Γ is minimal among the environments ρ such that $a \in [\![M]\!]\rho$.

PROOF.

By induction on the structure of M.

- M is a variable. The proof follows from the fact that $[\![M]\!]\rho_\Gamma = \rho_\Gamma(M)$.

- $M \equiv \lambda x.M'$. $\Gamma \vdash \lambda x.M'{:}b$ implies $b = i((\{a_1,\dots,a_n\},a))$, where $\Gamma \cup \{x{:}a_1,\dots,x{:}a_n\} \vdash M'{:}a$, $\{a_1,\dots,a_n\} \in D$ and $x \notin \mathrm{dom}(\Gamma)$. So, by induction $\rho' = \rho_\Gamma[\{a_1,\dots,a_n\}/x]$ is a minimal environment such that $a \in [\![M']\!]\rho'$. Since by definition we have both $[\![\lambda x.M']\!]\rho_\Gamma = K_i(\lambda d \in D.\ [\![M']\!]\rho_\Gamma[d/x])$ and $(\{a_1,\dots,a_n\},a) \in Tr(\lambda d \in D.\ [\![M']\!]\rho_\Gamma[d/x])$ we can conclude that $K_i(\{(\{a_1,\dots,a_n\},a)\}) = \{b\} \subset [\![\lambda x.M']\!]\rho_\Gamma$. The minimality follows from the definition of interpretation of a term. In fact if there were ρ'' such that $b \in [\![\lambda x.M']\!]\rho''$ and $\rho'' < \rho$ then $[\![M']\!]\rho''[\{a_1,\dots,a_n\}/x] = a$, contrary to the minimality of ρ'.

- $M \equiv PQ$. $\Gamma \vdash PQ{:}a$ implies $\Gamma \equiv \Gamma' \cup (\cup_{1 \leq j \leq n} \Gamma_j)$ and $\Gamma' \vdash P{:}b$, $(\Gamma_j \vdash Q{:}a_j)_{1 \leq j \leq n}$, $(\{a_1,\dots,a_n\},a) = i(b)$ $(n \geq 0)$. Since $[\![PQ]\!]\rho_\Gamma = H_i([\![P]\!]\rho_\Gamma)[\![Q]\!]\rho_\Gamma$, this implies $a \in [\![PQ]\!]\rho_\Gamma$. Moreover assume that there exists an environment $\rho' < \rho_\Gamma$ such that $a \in [\![PQ]\!]\rho'$. This implies $\exists \{b_1,\dots,b_m\}$ such that $(\{b_1,\dots,b_m\},a) = i(c)$ and $c \in [\![P]\!]\rho'$ and $b_j \in [\![Q]\!]\rho'$ $(1 \leq j \leq m)$. But, by Proposition 8.i), $\rho' < \rho_\Gamma$ implies $[\![Q]\!]\rho' < [\![Q]\!]\rho_\Gamma$, which means in particular that $\forall t (1 \leq t \leq n)\ \forall j\ (1 \leq j \leq m)$ a_t and b_j are compatible. But, by Proposition 3.ii), this means that $(\{a_1,\dots,a_n\},a)$ and $(\{b_1,\dots,b_m\},a)$ cannot approximate the same stable function, and hence b and c cannot belong to the interpretation of the same term.

□

COMPLETENESS THEOREM.

Let (D,i) be a qualitative λ-structure, and let M be a λ-term. Then $a \in [\![M]\!]\rho$ in (D, i) implies that there is a context Γ such that $\Gamma \vdash M{:}a$ and $\Gamma \subset \Gamma_\rho$, where $\Gamma_\rho(x)=\{x{:}b \mid b\in\rho(x)\}$.

PROOF.

By induction on the structure of M.

- $M\equiv x$. Then $a\in[\![x]\!]\rho$ implies $\rho(x)=A$ and $a\in A$. But $\{x{:}a\}\vdash x{:}a$ and $\{x{:}a\}\subset\Gamma_\rho=\{x{:}b \mid b\in A\}$.

-$M\equiv\lambda x.M'$. $a\in[\![\lambda x.M']\!]\rho$ implies $a=i((\{a_1,\dots,a_n\},b))$, where $b\in[\![M']\!]\rho[\{a_1,\dots,a_n\}/x]$ and $b\notin[\![M']\!]\rho[A/x]$ for any $A\subset\{a_1,\dots,a_n\}$ ($A\neq\{a_1,\dots,a_n\}$); this implies (by induction) that there is a context $\Gamma\subset\Gamma_{\rho[\{a_1,\dots,a_n\}/x]}$ such that $\Gamma\vdash M'{:}b$ and $\Gamma(x)=\{a_1,\dots,a_n\}$.

So we have $\Gamma-\{x{:}a_1,\dots,x{:}a_n\}\vdash\lambda x.M'{:}a$.

- $M\equiv PQ$. Then $a\in[\![PQ]\!]\rho$ implies $(\{a_1,\dots,a_n\},a)\in H_i([\![P]\!]\rho)$ and $a_j\in[\![Q]\!]\rho$ $(1\le j\le n)$ $(n\ge0)$. So, by induction, there exist contexts $\Gamma\subset\Gamma_\rho$ and $\Gamma_j\subset\Gamma_\rho$ such that: $\Gamma\vdash P{:}b$ such that $(\{a_1,\dots,a_n\},a)=i(b)$ and $\Gamma_j\vdash Q{:}a_j$ $(1\le j\le n)$, and this implies $\Gamma\cup(\cup_{1\le j\le n}\Gamma_j)\vdash PQ{:}a$. □

The following corollary summarizes the relationship between the qualitative λ-structure (D,i) and $S_{(D,i)}$.

COROLLARY 9.

$[\![M]\!]\rho=\{a \mid \exists\Gamma.\ \Gamma\vdash M{:}a \text{ and } \Gamma\subset\Gamma_\rho\}$.

The following proposition is an interesting consequence of the relevant discipline of assumptions in $S_{(D,i)}$.

Let M be a λ-term. FV(M) will denote the set of variables occurring free in M.

PROPOSITION 10.

i) Let $\Gamma\vdash M{:}a$. Then $x\in\mathrm{dom}(\Gamma)$ implies $x\in FV(M)$.

ii) Let $\Gamma\vdash M{:}a$ and $\Gamma'\vdash M{:}b$. If Γ and Γ' are consistent then $\{a,b\}\in D$, and moreover if $a=b$ then $\Gamma\equiv\Gamma'$.

PROOF.

i) Easy, by induction on M.

ii) The first part follows from the completeness theorem above and Proposition 8.ii). The second part is an easy induction on the structure of M taking into account Proposition 3. □

4. EXAMPLES OF TYPE ASSIGNMENT SYSTEMS CORRESPONDING TO PARTICULAR QUALITATIVE λ-MODELS.

In this section we will show how to tailor the formal system defined in the previous section to the qualitative λ-models Q, S and P introduced in Section 2. We shall denote them with $\vdash_Q$, $\vdash_S$, $\vdash_P$ respectively. Since each of these models has a nice finite inductive presentation we shall define $\vdash_Q$, $\vdash_S$, $\vdash_P$ as type assignment systems. The technique illustrated by these examples should be easily applicable to any inductively presented λ-model $M\equiv(D,i)$. Actually a large portion of the construction we are about to carry out will be common to the three type assignment systems. We shall specialize it only at the very end.

We start by giving finite inductive presentations of the models Q, S, P. This amounts to giving three formal systems F_Q, F_S, F_P. Let M be any of the three models Q, S, P; the system F_M will characterize the language of types and will play also the role of the systems $\in_D$ and $=_i$ of the previous section.

As a preliminary step we define the language L. Let $V=\{\phi_i \mid i\in\omega\}$ be an infinite set of variables. Terms of the language L, ranged over by α, are defined as follows:

$\alpha ::= \phi_1 \mid \phi_2 \mid \dots \mid [\alpha_1,\dots,\alpha_n]\to\alpha \mid [\,]\to\alpha \quad (n\ge1)$.

The systems F_M formalizes one unary judgement over L called **type** and four judgements over L×L, called **comp**, **compe**, **nonc** and **≈**. The intended meaning of these judgements, can be expressed using a surjective function I from terms α of the language L satisfying the judgment **type** α and atoms of the model $M=(D,i_M)$, as follows:

comp α,α' holds if and only if $I(\alpha)$ and $I(\alpha')$ are compatible but different atoms of M

compe α,α' if and only if $I(\alpha)$ and $I(\alpha')$ are compatible and possibly equal atoms of M

nonc α,α' if and only if $I(\alpha)$ and $I(\alpha')$ are incompatible atoms of M

$\alpha\approx\alpha'$ if and only if $I(\alpha)$ is equal to $I(\alpha')$.

$I([\alpha_1,\dots,\alpha_n]\to\alpha)=i_M((\{I(\alpha_1),\dots,I(\alpha_n)\},I(\alpha)))$

DEFINITION 11.

i) The following rules are common to the three systems F_Q, F_P, F_S:

$$1)\ \dfrac{\phi\in V}{\textbf{type}\ \phi}\qquad 2)\ (n\ge0)\ \dfrac{\textbf{type}\ \alpha\qquad (\textbf{comp}\ \alpha_i,\alpha_j)_{1\le i,j\le n}}{\textbf{type}\ [\alpha_1,\dots,\alpha_n]\to\alpha}\qquad 3)\ \dfrac{\textbf{comp}\ \alpha',\alpha}{\textbf{compe}\ \alpha',\alpha}$$

$$4)\ \dfrac{}{\textbf{compe}\ \alpha,\alpha}\qquad 5)\ \dfrac{\textbf{compe}\ \alpha,\alpha'}{\textbf{compe}\ \alpha',\alpha}\qquad 6)\ \dfrac{\textbf{comp}\ \alpha',\alpha}{\textbf{comp}\ \alpha,\alpha'}\qquad 7)\ \dfrac{\textbf{nonc}\ \alpha,\alpha'}{\textbf{nonc}\ \alpha',\alpha}$$

$$8)\ (n+m\ge1)\ \dfrac{(\textbf{comp}\ \alpha_i,\alpha_j)_{1\le i,j\le n+m}\qquad \alpha\approx\alpha'}{\textbf{nonc}\ [\alpha_1,\dots,\alpha_n]\to\alpha,\ [\alpha_{n+1},\dots,\alpha_{n+m}]\to\alpha'}$$

$$9)\ (n+m\ge0)\ \dfrac{(\textbf{comp}\ \alpha_i,\alpha_j)_{1\le i,j\le n+m}\quad \textbf{nonc}\ \alpha,\alpha'}{\textbf{nonc}\ [\alpha_1,\dots,\alpha_n]\to\alpha,\ [\alpha_{n+1},\dots,\alpha_{n+m}]\to\alpha'}$$

$$10)\ (n+m\ge0)\ \dfrac{\textbf{nonc}\ \alpha_1,\alpha'_1\qquad (\textbf{comp}\ \alpha_i,\alpha_j)_{1\le i,j\le n}\qquad (\textbf{comp}\ \alpha'_i,\alpha'_j)_{1\le i,j\le m}}{\textbf{comp}\ [\alpha_1,\dots,\alpha_n]\to\alpha,\ [\alpha'_1,\dots,\alpha'_m]\to\alpha'}$$

$$11)\ (n+m\ge0)\ \dfrac{(\textbf{compe}\ \alpha_i,\alpha_j)_{1\le i,j\le n+m}\quad \textbf{comp}\ \alpha,\alpha'}{\textbf{comp}\ [\alpha_1,\dots,\alpha_n]\to\alpha,\ [\alpha_{n+1},\dots,\alpha_{n+m}]\to\alpha'}$$

$$12)\ (n\ge1)\ \dfrac{\sigma \text{ is a permutation of } [1,\dots,n]\qquad \alpha\approx\alpha'}{[\alpha_1,\dots,\alpha_n]\to\alpha\approx[\alpha_{\sigma(1)},\dots,\alpha_{\sigma(n)}]\to\alpha'}$$

$$13)\ (n\ge0,\ m\ge1)\ \dfrac{[\alpha_1,\dots,\alpha_n]\to\alpha\approx[\alpha'_1,\dots,\alpha'_m]\to\alpha'\qquad \alpha\approx\alpha''}{[\alpha_1,\dots,\alpha_n]\to\alpha\approx[\alpha'',\alpha'_1,\dots,\alpha'_m]\to\alpha'}$$

14) $\dfrac{}{\alpha \approx \alpha}$ 15) $\dfrac{\alpha \approx \alpha'}{\alpha' \approx \alpha}$ 16) $\dfrac{\alpha \approx \alpha' \qquad \alpha' \approx \alpha''}{\alpha' \approx \alpha''}$

17) $\dfrac{\alpha \approx \alpha' \ \ \mathbf{type}\ \alpha}{\mathbf{type}\ \alpha'}$ 18) $\dfrac{\alpha \approx \alpha' \ \ \mathbf{comp}\ \alpha,\alpha''}{\mathbf{comp}\ \alpha,\alpha'}$ 19) $\dfrac{\alpha \approx \alpha' \ \ \mathbf{nonc}\ \alpha,\alpha''}{\mathbf{nonc}\ \alpha,\alpha'}$

20) $\dfrac{\alpha \approx \alpha' \ \ \mathbf{compe}\ \alpha,\alpha''}{\mathbf{compe}\ \alpha,\alpha'}$

ii) F_Q is obtained by adding the following rule to the system at point i):

$$\frac{\phi \in V \quad \mathbf{type}\ \alpha}{\mathbf{comp}\ \phi,\alpha}$$

iii) F_S is obtained by adding the following rules to the system at point i):

$$\frac{\phi_i, \phi_j \in V}{\phi_i \approx \phi_j} \qquad \frac{\phi_i \in V}{\phi_i \approx [\,] \rightarrow \phi_i}.$$

iv) F_P is obtained by adding the following rules to the system at point i):

$$\frac{\phi_i, \phi_j \in V}{\phi_i \approx \phi_j} \qquad \frac{\phi_i \in V}{\phi_i \approx [\phi_i] \rightarrow \phi_i}.$$

We denoted integer parameters in rules explicitly, as we did for the rules of $S_{(D,i)}$.

A few comments on this definition are in order. Rule 2 is a distinctive rule of binary qualitative domains. Rule 8 is a distinctive rule of the behaviour of traces of stable functions. The rules for $\approx$ imply that $\approx$ is a congruence relation, that the predicates are taken up to this congruence and that types are equivalent at least up to set-theoretical equality of sub-expressions of the shape $[\alpha_1,\ldots,\alpha_n]$. The remaining rules of i) should be self explanatory. Rules ii), iii), iv) can be justified on the basis of the initial morphisms respectively of Q, S and P, given at the end of Section 2. For instance the rule in ii) states that, if $Q=(D_Q,i_Q)$, where $D_Q=\lim_{n\geq 0} D_n$, atoms of D_0 are compatible with all types. Moreover these atoms are not functional and this implies that the model is not extensional. The first rule in iii) accounts for the fact that if $S=(D_S,i_S)$, where $D_S=\lim_{n\geq 0} D_n$, D_0 has a unique atom. The second rule enforces the equivalence between this unique atom and the atom in the trace of i_0, which implies that the model is extensional. The rules of iv) can be justified similarly.

If **type** α holds, then we will say that α is a type. The set of types will be ranged over by σ,τ. We are now ready to introduce the type assignment systems $\vdash_Q$, $\vdash_S$, $\vdash_P$.

DEFINITION 12.

i) A *basis* B is a set of pairs $x{:}\sigma$, where σ is a type. We will denote the set $\{x \mid x{:}\sigma \in B\}$ with dom(B).

ii) Let M be any of the models Q, S, P. Types are assigned to terms according to the following *type assignment system* $\vdash_M$, where $B \vdash_M M{:}\sigma$ denotes that M has type σ in M, under the assumptions recorded in B. The type assignment system $\vdash_M$ is

obtained by adding to the corresponding system F_M the following rules:

$$(\mathrm{var})\ \frac{}{\{x{:}\sigma\} \vdash_M x{:}\sigma}$$

$$(\rightarrow E)_{(n\geq 0)}\ \frac{B \vdash_M M{:}[\sigma_1,\ldots,\sigma_n]\rightarrow\sigma \qquad (B_i \vdash_M N{:}\sigma_i)_{1\leq i\leq n}}{B \cup (\cup_{1\leq i\leq n} B_i) \vdash_M MN{:}\sigma}$$

$$(\rightarrow I)_{(n\geq 0)}\ \frac{B \cup \{x{:}\sigma_1,\ldots,x{:}\sigma_n\} \vdash_M M{:}\sigma \qquad (\mathbf{comp}\ \sigma_i,\sigma_j)_{1\leq i,j\leq n} \qquad x \notin \mathrm{dom}(B)}{B \vdash_M \lambda x.M{:}\ [\sigma_1,\ldots,\sigma_n]\rightarrow\sigma}$$

$$(\approx_M)\ \frac{B \vdash_M M{:}\sigma \qquad \sigma \approx_M \tau}{B \vdash_M M{:}\tau}$$

$\approx_M$ denotes the congruence relation between types in F_M.

The first part of the following theorem states that the formal systems of Definition 11 are sound and complete syntactic descriptions of Q, S, P. The second part rephrases the results of Section 3 for Q, S, P and states that the type assignment systems above solve our problem for these models.

ISOMORPHISM THEOREM.

Let M be any of the models Q, S, P.

a) For each model $M=(D,i_M)$ there is a surjective function I_M from terms of L satisfying the judgment **type** α to the atoms of M, satisfying the following conditions:
   i) **compe** α,α' holds if and only if $I_M(\alpha)$ and $I_M(\alpha')$ are compatible and possibly equal atoms of M;
   ii) **comp** α,α' holds if and only if $I_M(\alpha)$ and $I_M(\alpha')$ are compatible but different atoms of M;
   iii) **nonc** α,α' holds if and only if $I_M(\alpha)$ and $I_M(\alpha')$ are incompatible atoms of M;
   iv) $\alpha \approx \alpha'$ holds if and only if $I_M(\alpha)$ and $I_M(\alpha')$ are equal atoms of M;
   v) $I_M([\alpha_1,\ldots,\alpha_n]\rightarrow\alpha) = i_M((\{I_M(\alpha_1),\ldots,I_M(\alpha_n)\}, I_M(\alpha)))$.

b) For each M the type system $\vdash_M$ is sound and complete for M, i.e.
   $[\![M]\!]^M\rho = \{I_M(a) \mid \exists B.\, B \vdash_M M{:}a \text{ and } B \subset B_\rho\}$, where $B_\rho(x)=\{b \mid I_M(b) \in \rho(x)\}$.

PROOF.

We will only outline this proof.

a) In the case of the model Q the isomorphism I_Q is inductively defined as follows:

$I_Q(\phi) = \psi_Q((\{\phi\},\varnothing))$ and

$I_Q([\sigma_1,\ldots,\sigma_n]\rightarrow\sigma) = \psi_Q((\varnothing,(\{I_Q(\sigma_1),\ldots,I_Q(\sigma_n)\},I_Q(\sigma))))$,

where $\psi_Q\colon |P(\omega)\times[D_Q\rightarrow_s D_Q]| \rightarrow |D_Q|$ is the characteristic isomorphism of Q. Thence $i_Q((\{I_Q(\sigma_1),\ldots,I_Q(\sigma_n)\},I_Q(\sigma))) = I_Q([\sigma_1,\ldots,\sigma_n]\rightarrow\sigma)$. The notation for the atoms of Q should be self-explanatory.

In the case of the model S the isomorphism I_S is inductively defined as: $I_S(\phi)=\psi_S((\varnothing,\phi))$ and $I_S([\sigma_1,\ldots,\sigma_n]\rightarrow\sigma) = \psi_S((\{I_S(\sigma_1),\ldots,I_S(\sigma_n)\}, I_S(\sigma)))$ where $\psi_S : |[D_S\rightarrow_s D_S]| \rightarrow |D_S|$ is the characteristic isomorphism of S.

In the case of the model P the isomorphism I_P is inductively defined as follows: $I_P(\phi)=\psi_P((\{\phi\},\phi))$ and $I_P([\sigma_1,\dots,\sigma_n]\rightarrow\sigma)=\psi_P((\{I_P(\sigma_1),\dots,I_P(\sigma_n)\},I_P(\sigma)))$, where $\psi_P:|[D_P\rightarrow_s D_P]|\rightarrow|D_P|$ is the characteristic isomorphism of P.

By a lengthy induction, taking into account the rules of F_M and the inductive nature of the atoms of M, one can now show that for each model the translation function I_M is well defined, surjective and satisfies the conditions i)-v). Therefore each M is isomorphic to the qualitative λ-model

$M'\equiv(\{A' \mid A'=\{\{\alpha\}\mid\exists A.A$ is a set of words in $L.\alpha\in A$ and $\forall\alpha,\beta\in A.$ (**type** α, **type** β and **comp** $\alpha,\beta))\}\},i_{M'})$

where we denote with $\{\sigma\}$ the equivalence class of each element σ of L modulo $\approx_M$ and where $i_{M'}((\{\{\sigma_1\},\dots,\{\sigma_n\}\},\{\sigma\}))=\{[\sigma_1,\dots,\sigma_n]\rightarrow\sigma\}$.

The obvious rewritings of the system $S_{(D_M,\, i_M)}$ and the soundness and completeness theorems allow us to prove part b).

□

The results of Section 3 can now be phrased in the language of type assignment systems. The proofs can be translated from the corresponding ones utilizing the isomorphisms given in the proof of the preceeding theorem. First of all we need to introduce the notion of coherent basis, in analogy to that of consistent context. A basis B is said to be *coherent* if $\{x:\sigma, x:\sigma'\}\subset B$ implies **comp** σ,σ'. Moreover two basis are said to be coherent if their union is coherent

PROPOSITION 13.
i) Let $B\vdash_M M:\sigma$. If $x\in dom(B)$ then $x\in FV(M)$.
ii) Let $B\vdash_M \lambda x.M:\sigma$ and let B be coherent. Then $\sigma=[\sigma_1,\dots,\sigma_n]\rightarrow\tau$ such that **type** σ.
iii) Let $B\vdash_M M:\sigma$, $B'\vdash_M M:\tau$ and let B and B' be coherent; then **comp** σ,τ.
iv) Let $B\vdash_M M:\sigma$, $B'\vdash_M M:\tau$, let B and B' be coherent and let $\sigma=\tau$; then $B=B'$.

□

5. TYPE ASSIGNMENT SYSTEMS AT WORK

In this section we will show how to derive properties of the λ-theory induced by the model M from properties of the type assignment system $\vdash_M$. Recall that the λ-theory induced by a model M is the set of equations:

$T_M=\{M=N \mid M,N\in\Lambda$ and $\forall\rho.\ [\![M]\!]^M\rho=[\![N]\!]^M\rho\}$.

We shall consider first the model Q.

First of all, we will show that every derivation Δ of $B\vdash_Q M:\sigma$ in $\vdash_Q$ is normalizable. Here normalizable means that Δ can be transformed into a derivation Δ' of $B\vdash_Q M':\sigma$ where no application of the rule $(\rightarrow I)$ in Δ' is immediately followed by an application of the rule $(\rightarrow E)$ and M' is a β-reduct of M. Using this fact we will show that the interpretation of a term in Q is the collection of the interpretations of its syntactical approximants. This will be called the Approximation Theorem for the model Q.

Trought this section, we will restrict ourselves to derivations where the basis is consistent.

DEFINITION 14.
i) Let Δ be the derivation: $B\vdash_Q M:\sigma$. A **cut** in Δ is an application of the rule $(\rightarrow I)$ immediately followed by an application of the rule $(\rightarrow E)$;
ii) The **degree** of a cut is the number of type symbols occurring in the premises of the application of the rule $(\rightarrow E)$ determining the cut.
iii) The **degree** of a derivation Δ, $G(\Delta)$, is the pair $\langle d,n\rangle$ where n is the number of cuts in Δ and d is the maximum degree of all cuts in Δ.
iv) A deduction Δ is **normal** if and only if $G(\Delta)=\langle 0,0\rangle$.

We consider the pairs ordered in lexicographic order (i.e., $\langle d,n\rangle\le\langle d',n'\rangle$ if and only if $(d\le d')$ or ($d=d'$ and $n\le n'$)).

LEMMA 15.
$\Delta:B\vdash_Q M:\sigma$ and $G(\Delta)>0$ implies that there exists Δ' such that $\Delta':B\vdash_Q M':\sigma$, where M β-reduces to M' and $G(\Delta')<G(\Delta)$.
PROOF.
We have to distinguish two cases.
1) At least one of the cuts with the maximum degree in Δ is of the following shape:

$$(\rightarrow E)\ \dfrac{(\rightarrow I)\ \dfrac{B\vdash_Q M:\tau \quad x\notin dom(B)}{B\vdash_Q \lambda x.M:[\,]\rightarrow\tau}}{B\vdash_Q(\lambda x.M)N:\tau.}$$

This implies that the full derivation is Δ: $B\cup B'\vdash_Q C[(\lambda x.M)N]:\sigma$, for a suitable context C[ ] and type σ.
Then, if x occurs in M at all, x occurs in subterms of M, $S_i[x]$ $(i\ge 0)$ say, which occur in subderivations of Δ of the shape:

$$\Delta_i:\quad (\rightarrow E)\ \dfrac{B_i'\vdash_Q R_i:[\,]\rightarrow\sigma_i}{B_i'\vdash_Q R_iS_i[x]:\sigma_i.}$$

Then Δ' is obtained from Δ by performing the following three operations:
i) replace every free occurrence of x with N in the subderivation ending with the cut. In particular this implies that every Δ_i is replaced by :

$$\Delta_i':\quad (\rightarrow E)\ \dfrac{B_i'\vdash_Q R_i:[\,]\rightarrow\sigma_i}{B_i'\vdash_Q R_iS_i[N\backslash x]:\sigma_i}$$

ii) replace $(\lambda x.M)N$ and every descendent of it with $M[N\backslash x]$
iii) delete the cut.
Thus we have Δ': $B\cup B'\vdash_Q C[M[N\backslash x]]:\sigma$ and $G(\Delta')<G(\Delta)$.
In the case x does not occur in M, simply erase the cut, and replace $(\lambda x.M)N$ and every descendent of it with M.
2) All the cuts with the maximum degree in Δ are of the following shape:

$$(\rightarrow E)\ \dfrac{(\rightarrow I)\ \dfrac{B\cup\{x:\sigma_1,\dots,x:\sigma_n\}\vdash_Q M:\tau \quad (\textbf{comp}\ \sigma_i,\sigma_j)_{1\le i,j\le n} \quad x\notin dom(B)}{B\vdash_Q\lambda x.M:[\sigma_1,\dots,\sigma_n]\rightarrow\tau} \qquad (B_i\vdash_Q N:\sigma_i)_{1\le i\le n}}{B\cup(\cup_{1\le i\le n}B_i)\vdash_Q(\lambda x.M)N:\tau.}$$

Pick one of these. Let the full derivation be Δ: $B\cup B'\vdash_Q C[(\lambda x.M)N]:\sigma$, for a suitable context C[] and type σ. Now, by Proposition 13.i), x occurs free in M. Let $p\ge 1$ be the number of occurrences in Δ of subderivations Δ_i $(1\le i\le p)$, consisting of an application of the (var) rule:

Δ_i : (var) $\dfrac{}{\{x:\sigma_i\} \vdash_Q x:\sigma_i}$.

Then Δ' is obtained from Δ by performing the following three operations:

i) replace every free occurrence of x with N in the subderivation ending with the cut. In particular this implies that every Δ_i is replaced by: $\Delta_i' : B_i \vdash_Q N:\sigma_i$ $(1 \le i \le n)$, and that the occurrences of x in subterms of M for which no type has been derived in Δ, if any, are handled as in 1)

ii) replace$(\lambda x.M)N$ and every descendent of it with $M[N\backslash x]$

iii) delete the cut.

Thus we have $\Delta' : B \cup B' \vdash_Q C[M[N\backslash x]]:\sigma$ and $G(\Delta') < G(\Delta)$.

□

The following theorem is an easy consequence of the lemma we have just proved.

THEOREM 16.
If $\Delta:B \vdash_Q M:\sigma$ then there exists a normal derivation Δ' and a term M' such that M β-reduces to M' and $\Delta':B \vdash_Q M':\sigma$.

□

We will now recall the notion of approximate normal form first introduced in [15] in order to discuss the interpretation of non-terminating λ-terms in Scott Domains.

DEFINITION 17.

i) The set $\boldsymbol{A}$ of the *approximate normal forms* is defined inductively as:
   - a term variable belongs to $\boldsymbol{A}$;
   - the constant Ω belongs to $\boldsymbol{A}$;
   - if $A_1,\dots,A_n$ belong to $\boldsymbol{A}$, then $\lambda x_1,\dots,x_m.zA_1\dots A_n$ belongs to $\boldsymbol{A}$, for any term variables $x_1,\dots,x_m,z$.

ii) If $M \in \Lambda$, the set of the *approximants* of M is:
   $A(M)= \{A \in \boldsymbol{A} \mid \exists M'.$ M β-reduces to M' and A and M' match up to subterms of M' corresponding to occurrences of Ω in A$\}$.

APPROXIMATION THEOREM.
$B \vdash_Q M:\sigma$ if and only if $\exists A \in A(M).\ B \vdash_Q A:\sigma$.
(i.e., $[\![M]\!]^Q\rho = \cup\{[\![A]\!]^Q\rho \mid A \in A(M)\}$).

PROOF.
(*only if* part) $B \vdash_Q M:\sigma$ implies (by Lemma 15) $\exists \Delta:B \vdash_Q M':\sigma$ and M β-reduces to M' and Δ is normal. Let A be the approximant of M obtained from M'by replacing with Ω every subterm of M' to which no type has been assigned by Δ. Clearly $B \vdash_Q A:\sigma$. If Δ has assigned a type to every subterm of M', then M' is in normal form and A=M'.
(*if* part) Let A be an approximant of M such that $\exists \Delta:B \vdash_Q A:\sigma$. Then M reduces to M', where M' is obtained from A by replacing the occurrences of Ω with suitable subterms, say $N_1,\dots,N_p$. Every occurrence of Ω in Δ must occur in subderivations of the shape:

$$(\to E)\ \dfrac{B' \vdash_Q A':[\,]\to\tau}{B' \vdash_Q A'\Omega:\tau.}$$

So a derivation $\Delta':B \vdash_Q M':\sigma$ can be obtained from Δ simply by replacing in Δ the occurrences of Ω with $N_1,\dots,N_p$ respectively. Since $[\![\ ]\!]^Q$ satisfies β-equality, by the Isomorphism Theorem, we have that $B \vdash_Q M':\sigma$ implies $B \vdash_Q M:\sigma$.

□

The Approximation Theorem is a powerful tool for investigating the theory induced by a

model. In this case it implies immediately, for instance, that the theory of the model Q is *sensible* (in the sense of [3]), and that $[\![Y]\!]$ is Tarski's least fixed point operator.
Moreover, using the Approximation Theorem and following the argument in [13], the theory of Q can be characterized by the following property.

PROPERTY 18.
$\forall\rho.\ [\![M]\!]\rho \sqsubseteq [\![N]\!]\rho$ if and only if
$\forall C[\ ]$.(if C[M] reduces to a head-normal-form with no initial abstractions then the same holds for C[N]).

The same property is satisfied by the interpretation in the filter model [2], which is the minimal solution to the domain equation $D = P(\omega)\times[D \to D]$, i.e., it is the limit model homologous to Q , constructed using Scott's Domains.. Moreover the theory of Q is the same as the theory of Scott's P_ω.

Here is an interesting corollary of the Approximation Theorem, for the λ-I-calculus.
The λ-I-calculus is the restriction of the λ-calculus to relevant terms.

DEFINITION 19.
A *relevant term* is inductively defined as follows:
- a variable is a relevant term;
- if M and N are relevant then MN is relevant;
- if M is relevant and $x \in FV(M)$ then $\lambda x.M$ is relevant.

Let a derivation Δ be *proper* if and only if no sub-expression of any type in Δ is [].

NORMALIZATION PROPERTY (for the λ-I-calculus).
Let M be a term of the λ-I-calculus. M is normalizable if and only if there exists a proper derivation $\Delta:B \vdash_Q M:\sigma$.

PROOF.
(*only if* part) Let M be a normal form of the λ-I-calculus. We can easily build a proper derivation Δ for M in the following way.
If $M \equiv x$, then $\Delta:\{x:\alpha\} \vdash x:\alpha$.
If $M \equiv \lambda x_1\dots x_n.zM_1\dots M_m$, let $\Delta_i:B_i \vdash M_i:\sigma_i$ $(1 \le i \le m)$ be proper derivations, such that Δ_i and Δ_j do not have type variables in common if $i \ne j$ (it is easy to see that it is always possible). Then one can show that $\cup_{1\le i\le m} B_i$ is a coherent basis. So Δ is the following proper derivation:

$$\{z:[\sigma_1]\to[\sigma_2]\to\dots\to[\sigma_m]\to\phi\} \vdash z:[\sigma_1]\to[\sigma_2]\to\dots\to[\sigma_m]\to\phi \quad B_1 \vdash M_1:\sigma_1$$

$$(\to E)\ \overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$\{z:[\sigma_1]\to[\sigma_2]\to\dots\to[\sigma_m]\to\phi\} \cup B_1 \vdash z\,M_1: [\sigma_2]\to\dots\to[\sigma_m]\to\phi \qquad B_2 \vdash M_2:\sigma_2$$

$$(\to E)\ \overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$\{z:[\sigma_1]\to[\sigma_2]\to\dots\to[\sigma_m]\to\phi\} \cup B_1 \cup B_2 \vdash z\,M_1M_2: [\sigma_3]\to\dots\to[\sigma_m]\to\phi$$

$$\vdots$$

$$(\to E)\ \overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$B' \equiv \{z:[\sigma_1]\to[\sigma_2]\to\dots\to[\sigma_m]\to\phi\} \cup (\cup_{1\le i\le m} B_i) \vdash zM_1\dots M_m:\phi$$

$$(\to I)\ \overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$B' - \{x_n:\tau_{n,1},\dots x_n:\tau_{n,p_n}\} \vdash \lambda x_n.\,zM_1\dots M_m:[\tau_{n,1},\dots,\tau_{n,p_n}]\to\phi$$

$$(\to I)\ \overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$\vdots$$

$$(\to I)\ \overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$B \equiv B' - (\cup_{1\le i\le n}\{x_i:\tau_{i,1},\dots x_i:\tau_{i,p_i}\}) \vdash \lambda x_1\dots x_n.zM_1\dots M_m:\sigma$$

where $\sigma \equiv [\tau_{1,1},\ldots,\tau_{1,p_1}] \to \ldots \to [\tau_{n,1},\ldots,\tau_{n,p_n}] \to \phi$, ϕ is a fresh type variable and $\{x_i{:}\tau_{i,1},\ldots x_i{:}\tau_{i,p_i}\}$ are all and only the assumptions on x_i in B'.

It is easy to see that, if M' is any term of the λ-I-calculus which is a β-expansion of M, then Δ can be transformed into a proper derivation $\Delta'{:}B'' \vdash M'{:}\sigma$.
(*if* part) In order to proof this part we need only to notice that if we apply repeatedly Lemma 15 to the derivation Δ: $B \vdash_Q M{:}\sigma$ we will reach eventually the normal form of M.
□

Finally we will give a result concerning the theory of the model ***P***. One can easily show that $\varnothing \vdash_P (\lambda x.xx)(\lambda x.xx) : \phi$ and hence $[\![(\lambda x.xx)(\lambda x.xx)]\!]^P \neq [\![\lambda x.(\lambda x.xx)(\lambda x.xx)]\!]^P$.

Hence the theory of ***P*** does not equate all unsolvable terms. One can actually show that this theory is not sensible, since $[\![I]\!]^P = [\![YB]\!]^P$ where I is the identity combinator, Y is the fixed point combinator and B is the composition combinator.
We shall end this paper with some conjectures concerning λ-theories induced by qualitative λ- models suggested by the nature of the type assignment systems $\vdash_M$. The theory of the model ***S*** should be the theory H^*, i.e. the maximal sensible theory and hence it should coincide with the theory of the homologous limit model constructed using Scott Domains. On the other hand the theory of the model ***P*** should be strictly included in the theory of the homologous limit model constructed using Scott Domains [10]. No qualitative λ-model should in fact induce such a theory.

ACKNOWLEDGMENTS.
The authors would like to thank M.Dezani-Ciancaglini for helpful discussions, and the referees for their useful comments.

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